Analysis of Asymmetric Stripline by Conformal Mapping

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Abstract—**This paper presents analysis of asymmetric stripline using a conformal mapping technique. The expression obtained for the characteristic impedance shows its functional dependence on the dimensions of the stripline. Equations for equipotential and flux lines and an expression for field distribution in the stripline are derived. Plots of impedance variation with strip width and strip displacement are presented.**

I. Introduction

A NUMBER of investigations on the formulation of impedance, potential, and field distributions of symmetric striplines have been reported in the literature. Some aspects of the striplines with an offset center conductor between the ground planes (asymmetric striplines) have been analyzed using numerical techniques. Kammler [1] has found values of capacitance for different displacements and widths of the center conductor using Green's function-integral equation technique. Mittra [2] has developed the modified residue calculus technique (MRCT) for finding charge and potential distributions.

In the present work, a conformal mapping technique is used to find the impedance potential and field distributions for an asymmetric stripline. Since the structure is asymmetric with respect to the plane of the strip, it is not possible to consider only one quadrant for the purpose of conformal mapping as was done by Collin [3]. Hence the transformation of one half of the structure which is symmetric with respect to the vertical plane is used in the present analysis. For this purpose the conformal transformation used by Yang and Lee [4] in connection with the transformation of two curved cylindrical plates into a rectangle has been employed. It is observed that a part of the expression for the transformation obtained by Yang and Lee [4] does not follow from the method of integration suggested in the literature [5]. The exact form of the expression is given in the present paper.

Application of the above transformation to the structure shown in Fig. 1 leads to a set of equations from which the structure required to satisfy the impedance requirements can be determined easily. The transformation involves elliptic integrals of the first and third kind. By separating the elliptic integrals with complex arguments into real and imaginary parts, exact equations for the equipotential and flux lines are obtained. An expression for the field distribution in the cross section of the stripline is derived from the complex potential function by taking the negative of the complex conjugate of its derivative. This transformation is more general in that it is applicable to a perfectly symmetric structure also. Variation of characteristic impedance with strip width and strip offset is evaluated. Distribution of the electric field at the top and bottom ground planes is determined for a particular case.

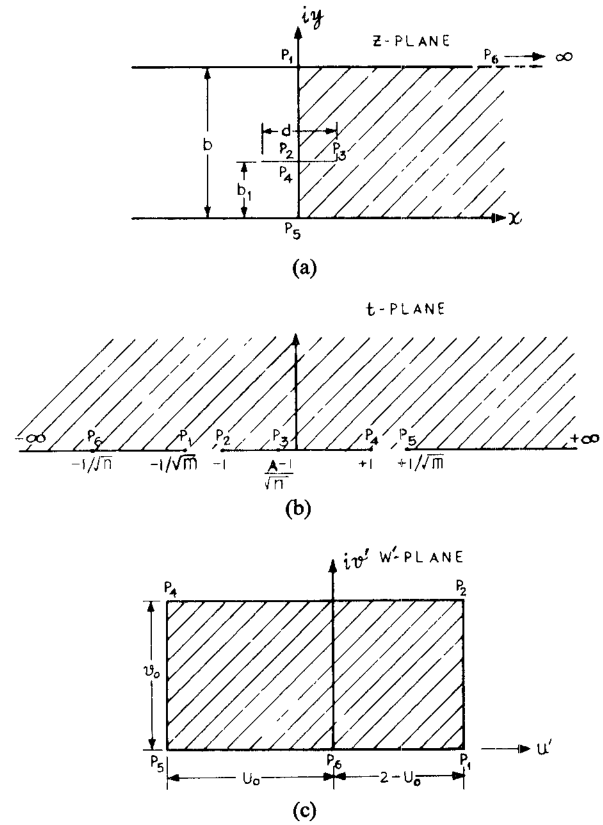

Fig. 1. An asymmetric stripline and its conformal representation.

II. Conformal Mapping

Consider the asymmetric stripline of Fig. 1(a). Ground planes are assumed to be infinitely wide, and the strip is assumed to have negligible thickness. The Schwarz–Christoffel transformation that transforms the upper half-plane of Fig. 1(b) (t-plane) into the shaded region of Fig.

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1(a) (z-plane) is given by [4]

$$Z=C_0\int_0^t \frac{\left(t+(1-A)/\sqrt{n}\right)dt}{\left(t+1/\sqrt{n}\right)\left[(t+1)(t-1)\left(t+1/\sqrt{m}\right)\left(t-1/\sqrt{m}\right)\right]^{1/2}}+B \tag{1}$$

where A, B, C_0, m, and n are constants and

$$0\leqslant n\leqslant m\leqslant 1 \text{ and } 0\leqslant\left|\frac{1-A}{n}\right|\leqslant 1.$$

Carrying on integration in terms of elliptic integrals and elliptic functions (1) takes the form

$$Z=C\left[u-A\left\{\Pi(n;u|m)-\sqrt{n}\,f(m,n,u)\right\}\right]+B \tag{2}$$

where $C=C_0\sqrt{m}$, $t=\sin\phi=\operatorname{sn}u$, and $u=F(\phi|m)$ is the incomplete elliptic integral of the first kind, and $\Pi(n;u|m)$ is the incomplete elliptic integral of the third kind [6], [7]. The exact expression for $f(m,n,u)$ derived by following the method suggested in the literature [5] is given by

$$f(m,n,u)=\frac{1}{2\sqrt{(1-n)(m-n)}}\cdot\ln\left\{\frac{2(1-n)(m-n)+(1-n\operatorname{sn}^2u)(n+nm-2m)}{n\left(2n-m-1+2\sqrt{(1-n)(m-n)}\right)(1-n\operatorname{sn}^2u)}+\frac{2n\sqrt{(1-n)(m-n)}\operatorname{cn}u\operatorname{dn}u}{n\left(2n-m-1+2\sqrt{(1-n)(m-n)}\right)(1-n\operatorname{sn}^2u)}\right\} \tag{3}$$

where sn u, cn u, and dn u are elliptic functions. Substituting the boundary conditions at the five points P_1, P_2, P_3, P_4, and P_5 into (2) and solving the resulting set of equations, the relations between the constants A, B, C, m, and n, and ground plane spacing b, strip width d, and the separation between the strip and the lower ground plane b_1 are obtained as

$$A=\frac{K(m)}{\Pi(n;K(m)|m)} \tag{4a}$$

$$B=\frac{b}{2\pi}\ln\left\{\frac{m-1}{2n-m-1+2\sqrt{(1-n)(m-n)}}\right\}+ib_1 \tag{4b}$$

$$C=-\frac{b}{\pi}\frac{\sqrt{(1-n)(m-n)}}{A\sqrt{n}} \tag{4c}$$

$$1-2\frac{b_1}{b}=\frac{F\left(\sin^{-1}\sqrt{n/m}\,|m\right)}{K(m)} \tag{4d}$$

$$\frac{d}{2b}=\frac{\sqrt{(1-n)(m-n)}}{\pi A\sqrt{n}}\left[F(\sin^{-1}\alpha|m)-A\Pi(n;\sin^{-1}\alpha|m)\right]-\frac{1}{2\pi}\ln\left[\frac{\sqrt{(1-n)(1-m\alpha^2)}-\sqrt{(m-n)(1-\alpha^2)}}{\sqrt{(1-n)(1-m\alpha^2)}+\sqrt{(m-n)(1-\alpha^2)}}\right] \tag{4e}$$

where $K(m)$ and $\Pi(n;K(m)|m)$ are complete elliptic integrals of the first and third kind, respectively, $\pi=3.14159$, and $\alpha=(1-A)/\sqrt{n}$.

The Schwarz–Cristoffel transformation that maps the upper half-plane of Fig. 1(b) (t-plane) into a rectangle in Fig. 1(c) (w'-plane) is given by [4]

$$W'=u'+iv'=C_1\int_0^t\frac{dt}{\sqrt{(1-t^2)(1-mt^2)}}+B_1 = C_1F(\phi|m)+B_1. \tag{5}$$

Evaluating the constants C_1 and B_1 from the boundary conditions at the points P_1, P_2, P_4, P_5, and P_6, the transformation is obtained as

$$W'=u'+iv'=-\frac{F(\phi|m)}{K(m)}-\frac{F\left(\sin^{-1}\sqrt{n/m}\,|m\right)}{K(m)}+\frac{iK'(m)}{K(m)}. \tag{6}$$

u_0 and v_0 shown in Fig. 1(c) are given by

$$u_0=1+\frac{F\left(\sin^{-1}\sqrt{n/m}\,|m\right)}{K(m)} \tag{7a}$$

$$v_0=K'(m)/K(m). \tag{7b}$$

In the limiting case, i.e., when the strip is placed exactly midway between the ground planes, the transformations become

$$Z=\frac{b}{2\pi}\ln\left\{\frac{m-1}{2m\operatorname{sn}^2u-m-1+2\sqrt{m}\operatorname{cn}u\operatorname{dn}u}\right\}+ib_1 \tag{8a}$$

and

$$W'=-\frac{F(\phi|m)}{K(m)}+\frac{iK'(m)}{K(m)}. \tag{8b}$$

The expression for strip width reduces to

$$d/b=\frac{1}{\pi}\ln\left\{\frac{1+\sqrt{m}}{1-\sqrt{m}}\right\} \tag{8c}$$

or $\sqrt{m}=\tanh(\pi d/2b)$ and is of the same form as that obtained by Collin [3].

III. Characteristic Impedance

One half of the structure shown in Fig. 1(a) is transformed into a parallel plate configuration as shown in Fig. 1(c). The width of the parallel plates is 2, and v_0 is the separation between them. The total capacitance per unit length of the stripline is twice that of the parallel plate capacitor in Fig. 1(c) and is given by

$$C'=\frac{4\epsilon_0\epsilon_r}{v_0}. \tag{9a}$$

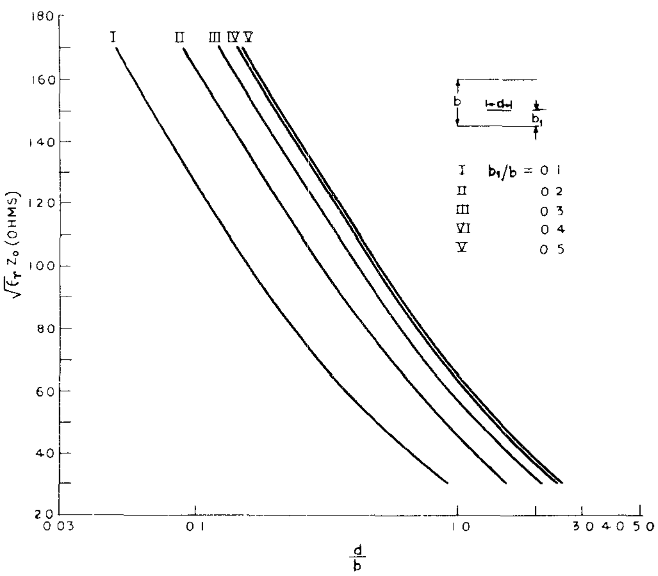

Fig. 2. Variation of characteristic impedance with strip width for different strip displacements.

The characteristic impedance of the stripline is given by

$$Z_0=\frac{30\pi}{\sqrt{\epsilon_r}}\cdot v_0$$
$$=\frac{30\pi}{\sqrt{\epsilon_r}}\frac{K'(m)}{K(m)}. \quad (9b)$$

The functional dependence of the characteristic impedance on the dimensions of the stripline is evident from (4a)–(4e), (9a), and (9b). For a given impedance of the line, the parameter m of the elliptic integrals is obtained from (9b). The characteristic n of the elliptic integral of the third kind is obtained from (4d). Remaining constants A, B, and C follow from (4a), (4b), and (4c), respectively. Finally, strip width is obtained from (4e). Variation of characteristic impedance with strip width for different displacements ($b_1/b=0.5$, 0.4, 0.3, 0.2, and 0.1) of the strip from the center is plotted in Fig. 2.

IV. Equipotential and Flux Lines

Since $t=\sin\phi$, ϕ is complex for any arbitrary point in the complex t-plane of Fig. 1(b). Separating the elliptic integral with complex argument in (6) into real and imaginary parts [7], the complex potential function is obtained as

$$W'=u'+iv'$$
$$=-\frac{F(\beta|m)-F(\sin^{-1}\sqrt{n/m}\,|m)}{K(m)}+i\frac{K'(m)-F(\gamma|m_1)}{K(m)} \quad (10a)$$

where $m_1=1-m$

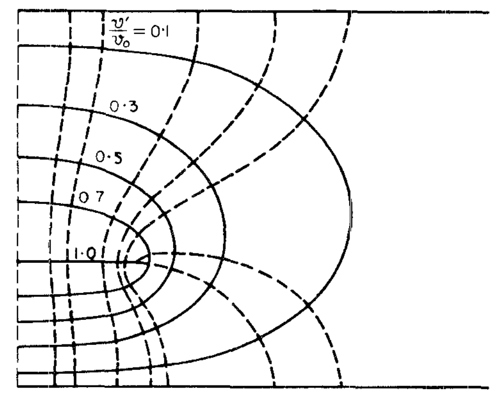

Fig. 3. Distribution of equipotential and flux lines for $b_1/b=1/3$ and $\sqrt{\epsilon_r}\, Z_0=80\ \Omega$. ——— Equipotential; - - - Fluxline.

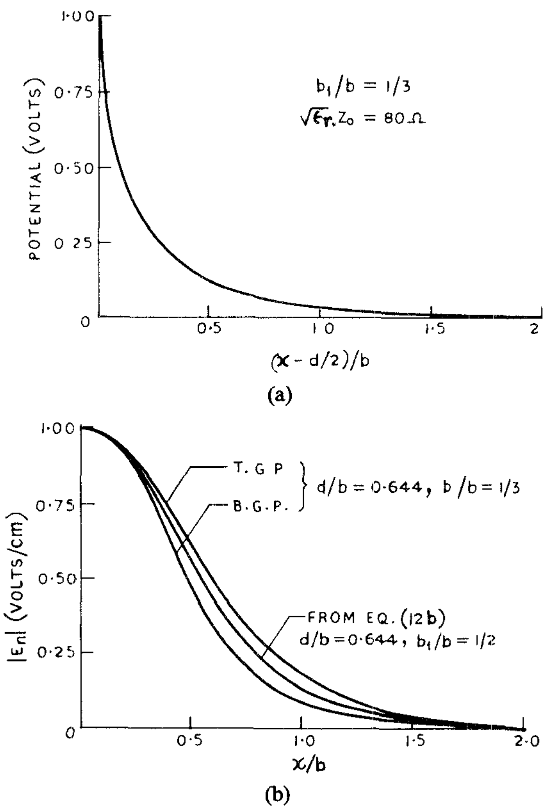

Fig. 4. (a) Potential distribution in the plane of the strip for $b_1/b=1/3$ and $d/b=0.644$. (b) Field distribution on the ground planes.

$$F(\eta\pm i\zeta|m)=F(\beta|m)\pm iF(\gamma|m_1)$$

$$\cosh\zeta\,\sin\eta=\frac{\sin\beta\sqrt{1-m_1\sin^2\gamma}}{\cos^2\gamma+m\sin^2\beta\,\sin^2\gamma}$$

$$\cos\eta\,\sinh\zeta=\frac{\cos\beta\,\cos\gamma\,\sin\gamma\sqrt{1-m\sin^2\beta}}{\cos^2\gamma+m\sin^2\beta\,\sin^2\gamma}.$$

The equation

$$u'=-\frac{1}{K(m)}\left[F(\beta|m)+F(\sin^{-1}\sqrt{n/m}\,|m)\right] \quad (10b)$$

represents the flux line, and the equation

$$v' = \frac{K'(m) - F(\gamma|m_1)}{K(m)} \tag{10c}$$

represents the equipotential line.

From the above equations, flux and equipotential lines are determined for $b_1/b = 1/3$ and $\sqrt{\epsilon_r}\, Z_0 = 80\ \Omega$ and are plotted in Fig. 3. Potential variation in the plane of the strip is presented in Fig. 4(a).

V. Expression for Field Distribution

The electric field in the cross section of the stripline is given by [8]

$$E = -\left(\frac{dW'}{dZ}\right)^* \tag{11a}$$

where the asterisk denotes a complex conjugate. Since the transformation of the stripline in Fig. 1(a) into the flux potential plane of Fig. 1(c) is obtained in two steps, (11a) becomes

$$E = -\left(\frac{dW'}{dt} \cdot \frac{dt}{dZ}\right)^*. \tag{11b}$$

From (2), (6), and (11b) the expression for the field distribution is obtained as

$$E = \frac{v}{CK(m)}\left[\frac{1}{1 - \dfrac{A}{1-nt^2} - \dfrac{nbt}{\pi C(1-nt^2)}\left\{\dfrac{2D_1\sqrt{1-t^2}\,\sqrt{1-mt^2} + \sqrt{D_1}\,(D_2 - D_3t^2)}{2D_1 + D_3(1-nt^2) + 2\sqrt{D_1}\,n\sqrt{1-t^2}\,\sqrt{1-mt^2}}\right\}}\right] \tag{12a}$$

where

$$D_1 = (1-n)(m-n)$$
$$D_2 = 2n - m - 1$$
$$D_3 = n + nm - 2m$$

and v is the potential difference between the strip and ground plane. In the perfectly symmetric case, the expression for the field reduces to

$$E = -\frac{\pi}{b\sqrt{m}\,K(m)}\left(\frac{1}{t}\right)^*. \tag{12b}$$

Thus the electric field in the cross section of the stripline has been obtained as an explicit function of the complex variable t, and its real and imaginary parts represent the vertical and horizontal components of the field, respectively. From (2)–(4) and (12), field distribution in the cross section of the line is evaluated. Normalized field distributions at the top and bottom ground planes calculated for $b_1/b = 1/3$ are presented in Fig. 4(b). In the same figure, field distribution at the ground planes for the perfectly symmetric case obtained from (12b) is also presented. The ratio of the maximum values of the electric field at the two ground planes as a function of the strip displacement, keeping the strip width constant ($d/b = 0.5$), is plotted in Fig. 5.

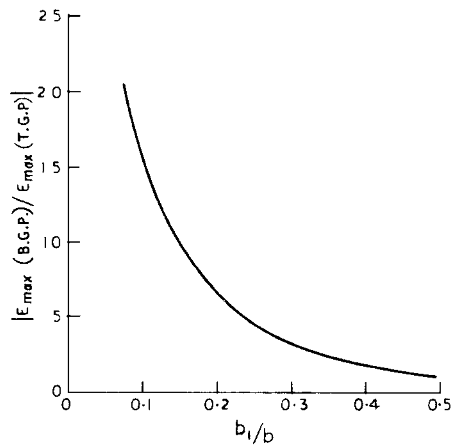

Fig. 5. Variation of the ratio of the maximum values of the field on the ground planes with strip displacement.

VI. Conclusions

The analysis presented here is quite general and can be used for the determination of impedance, potential, and field distributions of symmetric as well as asymmetric striplines. The analysis leads to exact equations for the equipotential and flux lines and enables one to calculate potential and field distribution from which the charge distribution on the strip can be easily evaluated. The rate of decay of the field, with distance from the plane of symmetry, is high, and the field decays to nearly 1 percent of its original value at a distance of about 1.6 times the ground-plane spacing.

The field distribution obtained is useful for the formulation of discontinuity problems [9] of apertures in the ground plane of asymmetric stripline. Investigations on the radiation properties of slots in the ground plane of a symmetric stripline reveal that there is appreciable loading on the line feeding the slot even when the slot is displaced [10]. This presents a problem in the design of slot array for narrow beamwidth. It is possible to reduce such loading by using a radiating slot in the farther ground plane of an asymmetric structure. The analysis on the field configuration of the asymmetric stripline, therefore, is of practical interest.

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Field and Network Analysis of Interacting Step Discontinuities in Planar Dielectric Waveguides

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Abstract—**Planar dielectric waveguides play an important role in electrooptics and at millimeter frequencies. In many laser configurations and integrated optical components, grooves are etched in the planar surface or overlays are deposited on it. The step is an idealization of such a discontinuity. Step discontinuities are seldom isolated. Mostly a cascade is employed. The aim of this paper is to derive, from a rigorous field analysis, an accurate finite network description for such cascades, either finite or infinite, periodic or aperiodic, which takes account also of the continuous spectrum. Numerical examples are given.**

I. INTRODUCTION

THE ANALYSIS of discontinuities in open dielectric waveguides is still in its infancy, and very few techniques are known [1]. In this paper we study an important class of discontinuities, namely, the cascade of steps in a planar dielectric waveguide, such as shown in Fig. 1. This is a basic configuration occurring in passive and active components for integrated optics and optical communications, such as the grating coupler, the transformer/echelon, and the distributed feedback laser. Corrugated dielectric waveguides are also used for millimeter waves and as microwave antenna feeds. Various approximations have been introduced for dealing with small discontinuities between monomode guides (see, for instance, [1]–[4]). The infinite periodic case has been treated extensively and rigorously (see, for instance, [5] for a most comprehensive list of references (287), as well as [6]). The problem of an isolated, large step between two multimode waveguides has been treated rigorously [7]. The general problem of arbitrarily large, aperiodic interacting steps is unsolved up to date. However, the optimum performance of various

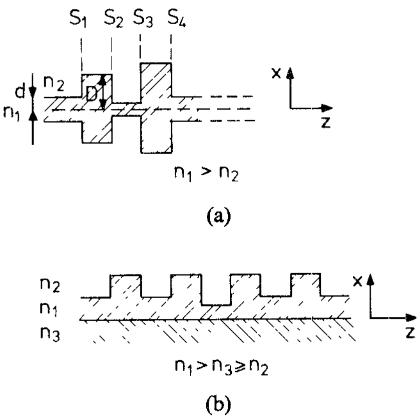

Fig. 1. Cascade of steps in a planar dielectric waveguide. (a) Cascade of symmetric steps. (b) Cascade of asymmetric steps.

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